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Interpolation Properties of Orthogonal Sets of Solutions of Differential Equations.[*]

By O. D. Kellogg.

As the present paper is of the nature of a continuation of the one published in the last number of this Journal, an extended introduction may be dispensed with. Suffice it to say that we shall here be concerned with the problem of extending to the orthogonal function sets arising from ordinary differential equations of second order, the properties there derived for sets arising from integral equations.

1. *The Differential Equation and Boundary Conditions.*

We shall be concerned with differential equations of the form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+(q(x)+\lambda r(x))y=0,\qquad p>0 \text{ for } 0<x<1,\quad r>0 \text{ for } 0\leqq x\leqq 1, \tag{1}$$

with the general homogeneous self-adjoint boundary conditions

$$ay(1)+by'(1)=cy(0)+dy'(0),\quad a'y(1)+b'y'(1)=c'y(0)+d'y'(0). \tag{2}$$

If $p(x)$ vanishes at one of the end points of the interval (0, 1), but in such a way that the differential equation has a solution which remains finite and different from zero, with a finite derivative in the neighborhood of this point, one of the boundary conditions (2) is to be replaced by the condition: $y(x)$ remains finite and different from 0 in the neighborhood of this point. If $p(x)$ vanishes at both end points, both equations (2) are to be replaced by this demand for both end points. Examples of these cases are: (a) the functions $J_0(a_0x)\sqrt{x}$, $J_0(a_1x)\sqrt{x}$, ..., where a_0, a_1, ... are the successive roots of the Bessel function of zero order $J_0(x)$; (b) the Legendre polynomials on the interval (—1, 1).

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The existence of solutions of the problem (1) and (2), and their oscillation properties have been extensively studied,[*] and we shall make use of some of the more common ones. Our interest will center rather in the interpolation properties, and some of the consequences of the property (D),[†] which it will be our object to establish.

By "harmonics" we shall understand solutions of the differential equation and boundary conditions. The corresponding "frequencies" are the values of λ for which these solutions are possible. There is an infinite sequence of frequencies, without finite limit point, and bounded below. We shall think of them written in ascending order of magnitude, a frequency being written twice if it corresponds to two harmonics. The harmonics are orthogonal;[‡] even two corresponding to the same frequency may be made orthogonal by a proper choice of the integration constants entering them. No harmonic vanishes simultaneously with its derivative. The zeros of any harmonic separate those of any harmonic with the same or lower frequency. In addition to these known facts, we assume explicitly (1) the continuity of the coefficients of the differential equation and of the derivatives involved, (2) that the boundary conditions are such that for any pair of harmonics

$$p(0)\begin{vmatrix}\phi_i(0), & \phi_j(0)\\ \phi_i'(0), & \phi_j'(0)\end{vmatrix} = p(1)\begin{vmatrix}\phi_i(1), & \phi_i(1)\\ \phi_j'(1), & \phi_j'(1)\end{vmatrix}, \tag{3}$$

and (3), that the i-th *harmonic has exactly* i *zeros in the interior of* $(0, 1)$ *for all* i. A harmonic is "even" or "odd" according as the number of interior zeros is even or odd.

2. *Types of Boundary Conditions.*

For what follows it will be convenient to separate the boundary conditions into types according to the behavior of the corresponding harmonics at the end points of $(0, 1)$. Consider first the case in which the determinant of coefficients in (2) $ab'-a'b$ vanishes. There will then be a relation of the form $a''\phi_i(0)+b''\phi_i'(0)=0$. From this it follows that the function

$$M(x)=p(x)\,[\phi_i'(x)\phi_j(x)-\phi_i(x)\phi_j'(x)]$$

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[*] See Bôcher, "Leçons sur les méthodes de Sturm," Borel monographs, Paris, 1917. Rich indications as to the literature are found in the footnotes.

[†] AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVIII, No. 1 (1916), particularly (1), (2), I, II, IV, V, VI and V′. Sturm (*Liouville's Journal*, Vol. I, p. 433), and Liouville (same Journal, Vol. I, p. 269) studied the interpolation problem, and the method of the latter has suggested that used here. But the only existing results appear to be confined to boundary conditions of Class I below, which does not include the periodic and other interesting cases.

[‡] Orthogonal here in the sense that $\int_0^1\phi_i(x)\phi_j(x)r(x)\,dx=0, \quad i \neq j$.

vanishes for $x=0$ for every i and j, and in consequence, by (3), also for $x=1$. We are thus led to the first class of boundary conditions, in which is also included the cases in which $p(x)$ vanishes at one or both end points.

Class I. $M(0)\equiv M(1)\equiv 0$. One boundary condition involves only one end point, and the other, only the other. Consider first the case in which $p(x)$ vanishes at an end point. Then under our assumption, all harmonics are different from 0 at that end point. Consider an end point at which $p(x)$ does not vanish, say $x=0$. Then, as $M(0)=0$, $\phi_i'(0)\phi_j(0)-\phi_i(0)\phi_j'(0)=0$, and since no harmonic vanishes simultaneously with its derivative, it follows that if any harmonic vanishes for $x=0$, all do. We conclude for Class I: *If any harmonic vanishes at an end point, all vanish there.*

Supposing the determinant $cd'-c'd=0$ leads to the same class. We may therefore assume that both this and $ab'-a'b$ are different from zero, and that $p(0)>0$ and $p(1)>0$. The conditions (2) may now be given the form

$$y(1)=ay(0)+by'(0), \qquad (4_1)$$
$$y'(1)=cy(0)+dy'(0), \qquad (4_2)$$

or

$$y(0)=dy(1)-by'(1), \qquad (5_1)$$
$$y'(0)=-cy(1)+ay'(1), \qquad (5_2)$$

where, by (3) and (4), $\Delta=ad-bc=p(0)/p(1)>0$. These conditions characterize the second class of cases.

Class II. $M(0)\equiv M(1)\not\equiv 0$. The boundary conditions both involve both end points. $M(0)$ and $M(1)$ may both vanish, but not as a consequence of the boundary conditions. There will be three types to consider in this class.

(a) $b=0$. Then $ad>0$, and (4_1) shows that if $\phi_i(x)$ vanishes at either end point, it does at the other. The sign of d now becomes important.

(a_1) $b=0$, $a>0$, $d>0$. If $\phi_i(x)$ vanishes at the end points, (4_2) shows that $\phi_i'(0)$ and $\phi_i'(1)$ have the same signs, so that $\phi_i(x)$ must have an odd number of interior zeros. But (4_1) shows the converse. Hence, for this case: *All odd harmonics vanish, and all even harmonics are different from zero at both end points.* An example of this case is $y''+\lambda y=0$, $y(1)=y(0)$, $y'(1)=y'(0)$, with the harmonics 1, $\sin 2\pi x$, $\cos 2\pi x$, $\sin 4\pi x$, $\cos 4\pi x$, ...

(a_2) $b=0$, $a<0$, $d<0$. By similar reasoning, we conclude for this case: *All even harmonics vanish, and all odd harmonics are different from zero at both end points.* An example is $y''+\lambda y=0$, $y(1)=-y(0)$, $y'(1)=-y'(0)$, with the harmonics $\sin \pi x$, $\cos \pi x$, $\sin 3\pi x$, $\cos 3\pi x$, ...

(b) $b>0$. Since a harmonic will not vanish simultaneously with its derivative, (4_1) shows that no harmonic vanishes at both end points. If

$\phi_i(0)=0$, (4_1) shows that $\phi_i(1)$ and $\phi'_i(0)$ have the same signs, and the harmonic must be even. The same conclusion follows from (5_1) when $\phi_i(1)=0$. Hence in this case: *No odd harmonic vanishes at either end point, and no harmonic vanishes at both.* An example is $y''+\lambda^2 y=0$, $\pi y(1)=2y'(0)$, $2y'(1)=-\pi y(0)$. The harmonics are, for $i\geqq 2$, $\phi_i(x)=\sin\lambda_i(x-1)+\frac{2}{\pi}\cos\lambda_i x$, where λ_i is the i-th positive root of $4\pi\lambda=(\pi^2+4\lambda^2)\sin\lambda$; while $\lambda_0=\lambda_1=\pi/2$. For these frequencies we have the solutions $A\cos\frac{\pi}{2}x+B\sin\frac{\pi}{2}x$. It will be seen if A and B are so chosen as to give an odd harmonic, this harmonic will be different from zero at both end points. All later harmonics are different from zero at both ends.

(c) $b<0$. By similar reasoning we conclude: *No even harmonic vanishes at either end point, and no harmonic does at both.*

3. *Lemmas.*

We shall need the following:

1. *If two harmonics have the same frequency, the one with the greater number of interior zeros is different from zero at both end points.* For the zeros of the two separate each other.

2. Let $y(x)=c_0\phi_0(x)+c_1\phi_1(x)+\ldots\ldots+c_n\phi_n(x)$ be a linear combination of harmonics, in which at least one of the constants c_i before the last is not 0. *If, for $c_n=0$, $y(x)$ has p interior roots, then for some $c_n\neq 0$, it will have at least p interior roots. If for $c_n=0$ it has an infinite number of interior roots, then for some $c_n\neq 0$, it will have at least N interior roots, N being any positive integer.* The proof is easily supplied by considering the limit of $y(x)$ as $c_n\doteq 0$, the approach of $y(x)$ to its limit being uniform. We shall reserve the notation $y(x)$ for functions of the above form.

3. *Given a function $y(x)$, $c_n\neq 0$, c_n remaining fixed, we can choose $c_0, c_1, \ldots, c_{n-1}$ so small, not necessarily zero, that $y(x)$ has the same number of interior zeros as $\phi_n(x)$ (i. e., exactly n), provided $\phi_n(x)$ does not vanish at an end point at which an earlier harmonic is different from zero. And in any case, $y(x)$ has not more than one additional interior zero corresponding to a terminal zero of $\phi_n(x)$.* The proof depends upon the facts that $\phi_n(x)$ does not vanish simultaneously with its derivative, and that all the harmonics are finite, with finite derivatives.

4. *Establishment of the Property* (D).

The results and treatment differ for the different types of boundary conditions enumerated. But the general argument is this. With a solution $\psi(x)$ of the differential equation, corresponding to boundary conditions to be assigned later, and not vanishing in the interior of $(0, 1)$, we form the differential operators to be applied to $y(x)$:

$$Ny(x)=p(x)\,[y'(x)\psi(x)-y(x)\psi'(x)], \text{ and } Ly(x)=\frac{d}{dx}Ny(x). \qquad (6)$$

It will first be shown that repeated application of the operator L reduces relatively the earlier coefficients of $y(x)$ to any desired degree, and thus, by the third lemma, leads to a function with no more than n interior roots. It will then be shown that the application of the operator L does not diminish the number of interior roots. The result, with the modifications noted, will be that no $y(x)$ has more than n interior roots, and accordingly, that the determinant of the equations $y(x_0)=0$, $y(x_1)=0, \ldots\ldots, y(x_n)=0$ does not vanish for any set of arguments x_i, no two of which are equal. The property (D) will follow (see this Journal, Vol. XL, p. 146, footnote).

Note first, however, that $D_0(x_0)>0$, since the first harmonic has no interior zero. Second, that in case $\lambda_1=\lambda_0$, $D_1(x_0, x_1)>0$. For the two harmonics, belonging to the same frequency, will have a non-vanishing Wronskian, from which fact the desired result may be inferred. We may therefore suppose in the following, $n\geqq 1$, or, in case $\lambda_1=\lambda_0$, $n\geqq 2$.

The choice of $\psi(x)$ in the operators (6) may vary with the function $y(x)$ to which it is to be applied, and the corresponding parameter value λ^* for which $\psi(x)$ is a solution of the differential equation (1) will vary with it. We shall assume, however, that there are two constants, $\varkappa$ and ρ, such that always

$$\varkappa\leqq\lambda^*\leqq\rho \text{ where } \rho<\frac{\lambda_0+\lambda_1}{2}, \text{ or if } \lambda_1=\lambda_0,\ \rho<\frac{\lambda_0+\lambda_2}{2}. \qquad (7)$$

The assumption will be considered below as use is made of it. Using the differential equation, it will be found that

$$Ly(x)=p(x)\psi(x)r(x)\,[c_0(\lambda^*-\lambda_0)\phi_0(x)+c_1(\lambda^*-\lambda_1)\phi_1(x) + \ldots\ldots+c_n(\lambda^*-\lambda_n)\phi_n(x)].$$

Dividing, on the assumption $c_n\neq 0$, by $p(x)\psi(x)r(x)\,(\lambda^*-\lambda_n)$ we arrive at a new function

$$y(x)=c_0\left(\frac{\lambda^*-\lambda_0}{\lambda^*-\lambda_n}\right)\phi_0(x)+c_1\left(\frac{\lambda^*-\lambda_1}{\lambda^*-\lambda_n}\right)\phi_1(x) + \ldots\ldots+c_{n-1}\left(\frac{\lambda^*-\lambda_{n-1}}{\lambda^*-\lambda_n}\right)\phi_{n-1}(x)+c_n\phi_n(x).$$

Under the assumption (7) the quantities $\left(\frac{\lambda^*-\lambda_i}{\lambda^*-\lambda_n}\right)$ are all less in absolute value than certain quantities independent of λ^* and less than 1, an exception arising only in the case of the last of them when $\lambda_n=\lambda_{n-1}$. Hence, by repeating the application of L and the division, we may reduce all the coefficients of $y(x)$ except the last (or, in case $\lambda_n=\lambda_{n-1}$, the last two) to be less than any assigned positive quantity. This completes our first step.

The next step is to study how this process affects the number of interior zeros in the various cases enumerated.

CLASS I. Choose $\psi(x)=\phi_0(x)$. Then $\lambda^*=\lambda_0$, and (7) is satisfied. If $y(x)$ has p interior zeros, $Ny(x)$ will be seen to vanish between each pair, and thus have at least $p-1$ interior zeros. But since in this case $M(x)$ vanishes at both end points for all pairs of harmonics, and as $y(x)$ is linear in the harmonics, it follows that $Ny(x)$ also vanishes at both end points, and hence $Ly(x)$ has at least p interior zeros. Thus no interior zeros are lost by the repeated application of L. Lemma 2 permits us to assume $c_n\neq 0$, and then Lemma 3 shows that no $y(x)$ has more than n interior zeros, since by § 2, no harmonics vanish at an end point under the present boundary conditions unless all do. And a case of double frequency does not arise.* *Hence, in problems of Class I, the property* (*D*) *holds.*

CLASS II, (a_1). $b=0$, $a>0$, and $y(1)=ay(0)$, for $y(x)$ is a linear homogeneous combination of harmonics, and hence satisfies the same boundary conditions. Hence $y(0)$ and $y(1)$, if not both zero, have the same signs. We again use $\psi(x)=\phi_0(x)$, and assume that the last harmonic in $y(x)$, $\phi_n(x)$ is even. If $y(x)$ has an odd number, $2p-1$, of interior zeros, it also has two at the end points, and $Ny(x)$ has therefore at least $2p$ interior zeros, and $Ly(x)$ at least $2p-1$. Thus, if the application of L to $y(x)$ changes this function from one with an odd number of interior zeros to a new one with an odd number, none are lost. If it changes $y(x)$ to a function with an even number of interior zeros, at least one is *gained.* If $y(x)$ has an even number, $2p$, of interior zeros, $Ny(x)$ has at least $2p-1$, but (3) shows that it then either has an additional interior zero, or else two at the end points, and in either case $Ly(x)$ has at least $2p-1$ interior zeros. If $Ly(x)$ has an even number of interior zeros, it must be at least $2p$, and if odd, at least $2p-1$. But ultimately, by Lemma 3, the repeated application of L leads to a function with an even number of interior zeros, since we have supposed this to be the case with

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* For two harmonics of the same frequency, $M(x)=$ const. This constant is not zero if the harmonics are independent. But it must be 0 in the problems of Class I.

$\phi_n(x)$. Hence no zeros have been lost, and we conclude: *The harmonics of Case II* (a_1) *have the property* (*D*) *for every even n.*

The conclusion is still justified in case $\lambda_{n-1}=\lambda_n$. Here we must study $c_{n-1}\phi_{n-1}(x)+c_n\phi_n(x)$. If this has an even number of interior zeros, their number can not exceed n, since they are separated by the zeros of $\phi_{n-1}(x)$, and $c_{n-1}\phi_{n-1}(x)+c_n\phi_n(x)$ will, by the boundary conditions, be different from 0 at the end points. The conclusion thus subsists in this case. If $c_{n-1}\phi_{n-1}(x)+c_n\phi_n(x)$ has an odd number of interior zeros, the repeated application of L to $y(x)$ may have reduced the number of interior zeros of $y(x)$ by one, but by no more, and $c_{n-1}\phi_{n-1}(x)+c_n\phi_n(x)$ has $n-1$ interior zeros, and none at the end points, since by § 2, $\phi_{n-1}(x)$ vanishes at both end points, and $\phi_n(x)$ at neither, and $c_n\neq 0$. The application of the lemma 3 then shows that $y(x)$ can not have more than n interior zeros, and the conclusion is established also in the case $\lambda_{n-1}=\lambda_n$.

Class II, (a_2). $b=0$, $a<0$, and $y(1)=ay(0)$, so that if $y(1)$ and $y(0)$ are different from zero, they have opposite signs. We now operate with L in which $\psi(x)$ is chosen subject to the condition that $y'(x)\psi(x)-y(x)\psi'(x)$ vanishes at both end points. By § 2, $\phi_0(x)$, being an even harmonic, vanishes at both end points, and as $\psi(x)$ is different from zero on the interior of $(0, 1)$, it follows that $\lambda^*\leqq\lambda_0$. Hence the assumption (7) is justified as far as the upper limit on λ^* is concerned. The existence of the lower limit will be generally established at the end of the paper. We start with a function $y(x)$ ending in an odd harmonic. The operator N does not reduce the number of interior zeros by more than one, while with the present choice of $\psi(x)$, $Ny(x)$ has zeros at the end points. So the operator L does not diminish the number of interior roots. As the odd harmonic $\phi_n(x)$, by § 2, is different from 0 at the end points, Lemma 3 gives the result: *The harmonics of Case II* (a_2) *have the property* (*D*) *for every odd n.*

Cases II, (b) and (c). Here, again, we choose $\psi(x)$ so as to make $Ny(x)$ vanish at the end points. It is not, in these cases, apparent that the upper limit for λ^* in (7) holds. However, since $\psi(x)$ does not vanish within $(0, 1)$, and as $\phi_2(x)$ vanishes twice, it is clear that $\lambda^*<\lambda_2$. As the frequencies have no finite limit point, there must be a positive integer n, and a number ρ, independent of λ^*, such that $\lambda^*\leqq\rho<\frac{\lambda_0+\lambda_n}{2}$. The least value of n for which these inequalities are possible,[*] we will call n'. The type of argument used in the previous cases then permits us to conclude: *The harmonics of Case II* (*b*) *have the property* (*D*) *for every odd* $n \geqq n'$, *and, provided none of the harmonics after* $\phi_{n'-1}(x)$ *vanishes at either end point, for every* $n \geqq n'$. *Also, the harmonics of Case II* (*c*) *have the property* (*D*) *for every even* $n \geqq n'$, *and, provided none of the harmonics after* $\phi_{n'-1}(x)$ *vanishes at either end point, for every* $n \geqq n'$.

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[*] This least value of n can often be easily determined in a particular problem, it being helpful to observe that the value of λ for which the differential equation has a solution vanishing at the end points of the interval (0, 1) but not in the interior, is an upper bound for λ^*. In the example in § 9 under Case II (b), (7) is fulfilled without alteration.

5. *Existence of a Lower Bound for* λ^*.

In the problems of Class II, from (a_2) inclusive, on, it was assumed that a lower bound for the values of λ^* existed. We proceed to show that such is the case. It will be recalled that $\psi(x)$ was to be chosen so that the ratios $\psi'(1)/\psi(1)$ and $\psi'(0)/\psi(0)$ coincided with the corresponding ratios for $y(x)$. Let us call these ratios η and ξ respectively. They are not independent, but are connected by the equation, obtained by dividing (4_2) by (4_1):

$$\eta=(c+d\xi)/(a+b\xi), \quad \text{where } ad-bc>0. \tag{8}$$

We shall now prove the theorem:

There exists a number Λ, *independent of* ξ *and* η, *such that the solution of the differential equation* (1) *which satisfies the boundary conditions*

$$u'(0)=\xi u(0), \quad u'(1)=\eta u(1), \tag{9}$$

subject to the relation (8), *and which does not vanish in the interior of* (0, 1), *belongs to a parameter value* $\lambda^*>\Lambda$.

We assume that $p(0)>0$ and $p(1)>0$. The theorem is not needed in the problems of Class I.

We first prove the lemma: *For fixed* ξ *and decreasing* η, *and for fixed* η *and increasing* ξ, *the value of the parameter* λ *corresponding to* (9) *increases.* Let $u_1(x)$ and $u_2(x)$ be two solutions of the differential equation satisfying the first equation (9). Denoting the corresponding parameter values by λ_1 and λ_2, we find the following identity:

$$p(1)[u_2(1)u_1'(1)-u_1(1)u_2'(1)]=(\lambda_2-\lambda_1)\int_0^1 u_2(x)u_1(x)r(x)\,dx.$$

We are, in this paragraph, concerned only with solutions of (1) which do not vanish in the interior of (0, 1), so that it is legitimate to consider $u_1(x)$ and $u_2(x)$ positive for $0<x<1$, so that the integral is positive. Dividing by $u_2(1)u_1(1)$, we have

$$p(1)[\eta(\lambda_1)-\eta(\lambda_2)]=\frac{(\lambda_2-\lambda_1)}{u_2(1)u_1(1)}\int_0^1 u_2(x)u_1(x)r(x)\,dx, \tag{10}$$

which shows that decreasing η corresponds to increasing λ. The second part of the lemma is proved similarly.

We next find the relation between ξ and η when these are the values of the ratios $u'(0)/u(0)$ and $u'(1)/u(1)$ corresponding to any *fixed* λ. To do this we take a fundamental solution set for $x=0$, $u_1(x)$ and $u_2(x)$, for which $u_1(0)=1$, $u_1'(0)=0$, $u_2(0)=0$, $u_2'(0)=1$, in terms of which any solution $u(x)$ for the same value of λ may be expressed in the form $u(x)=c_1\,u_1(x)+c_2\,u_2(x)$. Then $\xi=c_2/c_1$, and $\eta=[c_1u_1'(1)+c_2u_2'(1)]/[c_1u_1(1)+c_2u_2(1)]$. Eliminating the constants, we find the relation

$$\eta=\frac{u_1'(1)+\xi u_2'(1)}{u_1(1)+\xi u_2(1)}, \tag{11}$$

for which we have the easily derived relation

$$u_1'(1)u_2(1)-u_1(1)u_2'(1)=p(0)/p(1)>0.$$

This shows that the hyperbola which is the Cartesian representation of the equation (11) is an always rising one. For various values of λ, this hyperbola varies in position and size, but if for a finite value Λ of λ, its center lies above and to the left of the hyperbola (8), also an always rising hyperbola, the lemma shows that the solution of the differential equation and the boundary conditions (9) subject to (8) will always correspond to a value of $\lambda>\Lambda$, and the theorem will be proved.

That for some finite λ the center of the hyperbola (11) lies above and to the left of the upper left-hand branch of the hyperbola (8), will be evident from the fact which we shall next prove: given any finite positive quantities h and k, a finite λ always exists for which the coordinates of the center of the hyperbola (11) satisfy the inequalities

$$u_1(1)/u_2(1)>h, \quad u_2'(1)/u_2(1)>k. \tag{12}$$

The identity $\dfrac{u_2(1)}{u_1(1)}=\displaystyle\int_0^1\frac{p(0)}{p(x)u_1^2(x)}\,dx$, the derivation of which gives no trouble if the initial values of the fundamental solution set are kept in mind, shows that the ratio $u_1(1)/u_2(1)$ can be made as great as we wish if $u_1(x)$ can be made as great as we please at all interior points. But this fact can be inferred from the identity

$$\frac{d}{dx}\left(\frac{u_1(x,\lambda)}{u_1(x,\lambda_1)}\right)=\frac{\lambda_1-\lambda}{p(x)u_1^2(x,\lambda_1)}\times\int_0^x u_1(x,\lambda)u_1(x,\lambda_1)r(x)dx.$$

This shows that $u_1(x,\lambda)/u_1(x,\lambda_1)$ increases with x if $\lambda<\lambda_1$. This ratio approaches 1 as $x \doteq 0$. Hence always, $u_1(x, \lambda) > u_1(x, \lambda_1)$ for $0 < x < 1$, and using the law of the mean, $\frac{d}{dx}\frac{u_1(x, \lambda)}{u_1(x, \lambda_1)} > \frac{\lambda_1 - \lambda}{\text{max. } p}\int_0^x r(x)\,dx > (\lambda_1 - \lambda) \times$ a positive function of x independent of λ. Hence the ratio, and therefore $u_1(x, \lambda)$ can be made as large as we please, and the first inequality (12) is established.

To prove the second, we start by noticing that for sufficiently small λ, $u_2'(x) > 0$ for $0 < x < 1$. This is seen from the equation $p(x)u_2'(x) = p(0) + \int_0^x (-\lambda r - q)\,u_2\,dx$, in which the integral is positive if $\lambda < -\text{max. } q/\text{min. } r$. Multiplying the differential equation for u_2 by $2p(x)u_2'(x)$ and integrating, we have $[pu_2'(x)]^2 \Big|_0^1 = -2\int_0^1 p(\lambda r + q)\,u_2 u_2'\,dx = -\overline{p(\lambda r + q)}\,u_2^2(1)$, the bar denoting a mean value. Hence $u_2'(1)/u_2(1) > \frac{1}{p(1)}\sqrt{(-\lambda)\ \text{min. } pr - \text{max. } pq}$, which can be made as large as we please by sufficiently diminishing λ. Thus the second inequality (12) is proved.

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